

# Topologically Derived Separation Conditions for Two- and Three-Dimensional Laminar Flows

Murray Tobak\*

NASA Ames Research Center,

Moffett Field, CA 94035-1000

Topological concepts are used to derive separation conditions for two- and three-dimensional laminar flows. The result for two-dimensional flow reproduces the form of the well-known Stratford criterion. An extension makes the form applicable to the symmetry plane of a three-dimensional flow.

## Nomenclature

$p$	= pressure
$x, y, z$	= coordinate system
$u, v, w$	= velocity components
$\omega, \omega_3$	= vorticity component normal to xy plane
$\mu$	= fluid viscosity
$\rho$	= fluid density

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\* Senior Staff Scientist, MS 260-1, NASA Ames Research Center, Moffett Field, CA, 94035-1000. Associate Fellow AIAA

## I. Introduction

Recently, topological concepts were used to interpret results of water-channel experiments<sup>1</sup> and computations<sup>2,3</sup> for the laminar juncture-flow problem, which involves the flow approaching a cylindrical obstacle mounted on a flat plate. Confirmation of the presence of a saddle point of attachment, rather than one of separation, proved that the flow approaching the obstacle was able to find an alternative to separation in the classical sense. In the course of studying how it happened that a saddle point of attachment had appeared, it was found convenient to begin with virtually two-dimensional flow conditions by imagining the cylinder diameter to be very large. The flow approaching the cylinder necessarily had to separate, originating from a saddle point of separation at the wall, located on the symmetry plane of the approaching flow. Successive reductions in the diameter of the cylinder brought into play the transverse velocity component. It was possible to follow its influence on the evolution of the singular points and loci of zeroes of the velocity-vector components in the symmetry plane. The evolution led eventually to a critical condition at which the singular point in the external flow merged with the singular point at the wall, changing the saddle point of separation to one of attachment.

The idea of following the evolution of loci of zeroes of vector components may have wide application. It should prove useful, for example, in studying how

a two-dimensional separated flow becomes three-dimensional in response to the introduction of an initially infinitesimal control parameter that brings into play the third velocity component. The juncture-flow problem described above is one such example, where cylinder curvature was the control parameter.

Another example is the evolution with angle of attack of the initially axisymmetric separation bubble on the nose of a hemisphere-cylinder body, angle of attack being the initially infinitesimal control parameter. The idea also may prove helpful in the study of the flow approaching the leading-edge stagnation line of an initially two-dimensional obstacle. Here, it is possible to envision a sequence of flows directly analogous to that of the juncture-flow problem described above. In the analysis to follow, however, the idea is exploited in yet another direction. Following the evolution of loci of zeroes of vector components in the presence of an adverse pressure gradient leads to a concise derivation of critical conditions for the onset of separation. Separation conditions derived for a two-dimensional flow are found to reproduce the well-known Stratford criterion. A simple modification of the Stratford form is found applicable to the symmetry plane of a three-dimensional flow.

## II. Analysis of Sequence of Flows Leading to Two-Dimensional Separation

The analysis to follow applies to steady flows only. The flows will be laminar and incompressible, hence governed by the time-independent Navier-Stokes equations for incompressible flow. Let us begin with a fully attached flow over a smooth airfoil having a favorable pressure gradient. Let  $x$  represent distance along the wall (starting from zero at the leading-edge stagnation point),  $y$  represent distance normal to wall, positive into the flow. Consider the upper surface. As noted on Figure 1, since vorticity at the wall must begin and end with zero at the stagnation points, there will be a point on the surface where  $\omega_x = 0$ . At the wall,  $p_y = \mu\omega_x$ , so this is also a point where  $p_y = 0$ . On the other hand,  $p_x = 0$  at the stagnation points. Since at the wall,  $p_x = -\mu\omega_y$ ,  $\omega_y$  must equal zero at these points as well. Going around the airfoil, one notes that the points where  $p_x = 0$  alternate with the points where  $p_y = 0$ . Since  $\omega_x = 0$  at points where  $p_y = 0$ , and  $\omega_y = 0$  at points where  $p_x = 0$ , the extrema in wall vorticity also must alternate in a circuit around the airfoil. Although the proof will not be presented here, it is possible to show that this "alternation property" of the extrema in wall pressure and wall vorticity must hold as a general rule for all two-dimensional obstacles defined by smooth closed curves (i.e., curves transformable into a circle).

Now let us impose a region of adverse pressure gradient on the upper surface between the leading edge and the point where  $\omega_x = p_y = 0$  (Fig. 2).

Consider the domain of  $x$ ,  $a < x < b$  where  $p_x(x, 0) > 0$ . There are two points  $(x = a, b)$  where  $p_x(x, 0) = 0$ . Since at the wall  $p_x = -\mu\omega_y$ ,  $\omega_y = 0$  at  $x = a, b$ . The situation is sketched in Fig. 3. One notes that:

Outside  $a, b$ ,  $\omega_y(x, 0) > 0$ .

Within  $a, b$ ,  $\omega_y(x, 0) < 0$ .

At any  $x$ ,  $\omega_y > 0$  as  $y \rightarrow \infty$ ,

while for  $a < x < b$ ,  $\omega_y(x, 0) < 0$ .

So, for any  $x$ ,  $a < x < b$ , there must be a value of  $y > 0$  for which  $\omega_y = 0$ . There is such a  $y$  for every value of  $x$  between  $x = a$ ,  $x = b$ . Since  $\omega_y$  is a continuous function of  $(x, y)$ , there must be a curve,  $y = f(x)$  on which  $\omega_y = 0$ , connecting the points  $a, b$  (Fig. 4). Existence of a curve on which  $\omega_y = 0$  in the flow implies a domain where the velocity profile  $u(y)$  will be inflexional. Since  $u_y \approx -\omega$ ,  $u_{yy} \approx -\omega_y$  and  $u_{yy} = 0 \Rightarrow$  inflexion point.

The region of adverse pressure gradient implies existence of a curve  $y = f(x)$  on which  $\omega_y = 0$ . Existence of  $y = f(x)$  forces existence of a curve  $y = g(x)$  on which

$\omega_x = 0$  in order to fulfill the condition that points on the wall where  $\omega_y = 0$  must alternate with points where  $\omega_x = 0$ . The situation is sketched on Fig. 5. The "alternation property" also makes it necessary that the curve  $y = g(x)$  on which  $\omega_x = 0$  cross the curve  $y = f(x)$  on which  $\omega_y = 0$ . So there is a singular point in the flow at which  $\omega_x$  and  $\omega_y$  are simultaneously zero. It is possible to show that this singular point must be a saddle point.

Similarly, there will be a curve on which  $p_y = 0$  starting from the same origins as the  $\omega_x = 0$  curve, and a curve on which  $p_x = 0$  starting from the same origins as the  $\omega_y = 0$  curve. Similarly, the  $p_x = 0$  and  $p_y = 0$  curves must cross in the flow, giving a point where  $p_x$  and  $p_y$  are simultaneously zero. This singular point also must be a saddle point.

So the adverse pressure gradient creates extrema in pressure  $p$  and vorticity  $\omega$  in the flow, preceding the onset of separation; i.e., preceding the appearance of a singular point in the external flow. Therefore, just as surface pressure extrema precede and so must accompany the appearance of singular points in the skin-friction line pattern (cf. Ref. 4), it is already assured that singular points in the external velocity field are preceded and so must be accompanied by pressure and vorticity extrema.

The phase portrait of  $\text{grad}\omega \begin{pmatrix} \omega_x \sim u \\ \omega_y \sim v \end{pmatrix}$  is illustrated in Fig. 6. The critical point where separation will occur is where  $\omega_x(x,0) = 0$ ,  $|\omega| = \text{minimum}$  with respect to  $x$ . Separation will begin when the minimum value becomes identically zero. It is important to note that the curves of  $\text{grad}\omega$  must be orthogonal to the contour curves on each of which  $\omega = \text{constant}$ , and that the contour curves encircle the point where separation will begin.

### III. Separation Condition for Two-Dimensional Plane Flow

The phase portrait of  $\text{grad}\omega$  (Fig. 6) describes conditions at the onset of separation. For simplicity, let us assume that the wall is virtually plane in the vicinity of the point where separation will begin, so that  $x$  and  $y$  can be considered Cartesian. Shift the origin of  $x$  to the point where separation will begin; i.e., where  $\omega_x = 0$  on Fig. 6. With total head  $H$  defined as

$$H = p + \frac{1}{2}\rho(u^2 + v^2) \quad (1)$$

the Navier-Stokes equations take the form

$$\left. \begin{aligned} H_x &= -\mu\omega_y + \rho v\omega \\ H_y &= \mu\omega_x - \rho u\omega \end{aligned} \right\} \quad (2)$$

where  $\omega_y = -\nabla^2 u, \omega_x = \nabla^2 v$ . Introduce the stream function  $\Psi$  such that

$$\left. \begin{aligned} \Psi_x &= v \\ \Psi_y &= -u \\ \nabla^2 \Psi &= \omega \end{aligned} \right\} \quad (3)$$

Consider a curve  $y = q(x)$  and form  $dH$  along the curve:

$$dH = H_x dx + H_y dy = (H_x + H_y (x, q(x)) q'(x)) dx \quad (4)$$

Substituting (2) and (3), (noting that  $d\Psi = (\Psi_x + q'(x)\Psi_y)dx$  on  $y = q(x)$ ) yields

$$dH = \rho \omega d\Psi - \mu (\omega_y - q'(x)\omega_x) dx \quad (5)$$

on  $y = q(x)$ . If  $q(x)$  is allowed to be a closed curve consisting of the segments  $C, L_1, L_2$  (cf. Fig. 7), then integrating  $dH$  around  $q(x)$  must yield zero; i.e.,

$$\int_C dH + \int_{\downarrow L_1} dH + \int_{\leftarrow L_2} dH = 0 \quad (6)$$

$$\text{But } \int_{L_2} dH = \int_0^a p_x dx = -(p(0,0) - p(a,0)) = -\Delta p_w \text{ (from } a \text{ to } 0) \quad (7)$$

$$\text{so } \Delta p_w = \int_C dH + \int_{L_1} dH \quad (8)$$

From the phase portrait of  $\text{grad}\omega$  (Fig. 6), it can be seen that there is a contour curve on which  $\omega = \text{const}$ , originating at the singular point  $(x_0, y_0)$  where  $(\omega_y = 0, \omega_x = 0)$ , and ending at the wall at  $x = a$ . Let this curve be  $C$ . Then on  $C$ ,  $\omega = \text{const} = \omega_w(a)$  and

$$\int_C dH = \rho \omega_w(a) \int_C d\Psi - \mu \int_C (\omega_y - q' \omega_x) dx \quad (9)$$

$$= \rho \omega_w(a) \Psi(x_0, y_0) - \mu \int_C (\omega_y - q' \omega_x) dx \quad (10)$$

The second term in (10), the viscous contribution, can be omitted on the basis of the following argument: On  $C$ ,  $d\omega = 0$ , or  $\omega_x + q'(x)\omega_y = 0$ . Then the integrand  $\omega_y - q'(x)\omega_x$  becomes  $\omega_y(1 + q'^2)$ . But the curve  $y = q(x)$  on the segment  $C$  is very close to the curve  $y = f(x)$  on which  $\omega_y = 0$  (cf. Fig. 6), so that  $\omega_y(x, q(x)) \approx \omega_y(x, f(x)) = 0$ . This ensures that the viscous contribution to (10) will be negligibly

small. On the segment  $L_1$ ,  $L_1$  can be taken to be  $y = g(x)$  on which  $\omega_x = 0$ . Since  $g(x)$  is close to vertical,

$$\int_{\downarrow L_1} dH = \rho \int_{y_0}^0 \omega \frac{d\Psi}{dy} dy + \mu \int_{y_0}^0 \omega_x dy \quad (11)$$

But on  $L_1$ ,  $\omega \approx -u_y$ ,  $\Psi y \approx -u$ ,  $\omega_x = 0$ , so

$$\int_{\downarrow L_1} dH = \rho \int_{y_0}^0 u u_y dy = -\rho \frac{u^2(y_0)}{2} \quad (12)$$

Summing contributions yields

$$\Delta p_W = \rho \omega_w(a) \Psi(x_o, y_o) - \rho \frac{u^2(x_o, y_o)}{2} \quad (13)$$

This form in effect reproduces Stratford's principal result, arrived at in Ref. 5 by a heuristic argument (cf. also Ref. 6 for a more accessible account). From this point on, it is possible to follow Stratford's original analysis. The following conditions exist at  $x \approx x_o = 0$ :

At  $y = 0$ :

$$\left. \begin{aligned} u &= 0 \\ \mu u_{yy} &= p_x(0, 0) \end{aligned} \right\} \quad (14)$$

and  $u_y = 0$  at onset of separation.

At  $y = y_0$ :

$$\left. \begin{aligned} u_y &= u_y(a, 0) = -\omega_w(a) \\ u_{yy} &\approx 0 \\ \Psi(0, y_0) &= \frac{1}{\rho \omega_w(a)} \left\{ \Delta p_w + \frac{\rho u^2(0, y_0)}{2} \right\} \end{aligned} \right\} \quad (15)$$

$$\text{with } \Psi = -\int_0^y u(0, y_1) dy_1$$

Let

$$u = \alpha y + \beta y^2 + \gamma y^3 \quad (16)$$

and fit the constants to (14) and (15):

$$\left. \begin{aligned} \beta &= \frac{1}{2\mu} p_x(0,0) \\ \gamma &= -\frac{1}{6\mu y_0} p_x(0,0) \\ \alpha &= -\omega_w(a) \left( 1 + \frac{y_0}{\mu\omega_w} p_x(0,0) \right) \end{aligned} \right\} \quad (17)$$

$$\text{Let } \xi = -\frac{y_0}{\mu\omega_w} p_x(0,0) \quad (18)$$

$$\text{so } \alpha = -\omega_w(a) \left( 1 - \frac{\xi}{2} \right) \quad (19)$$

and the condition for separation requires  $\xi = 2$ .

Forming  $\Psi(0, y_0)$  and  $u(0, y_0)$  in terms of the constants results in:

$$\omega_w^2 y_0^2 \left( \frac{\xi}{3} \right) \left( 1 - \frac{\xi}{3} \right) = \frac{8\Delta p_w}{\rho} \quad (20)$$

$$\text{But from (18), } y_0^2 = \xi^2 \left( \frac{\mu\omega_w}{p_x} \right)^2$$

$$\text{so that } \omega_w^2 \left( \frac{\rho v \omega_w}{p_x} \right)^2 \frac{\xi^3}{3} \left( 1 - \frac{\xi}{3} \right) = \frac{8\Delta p_w}{\rho} \quad (21)$$

with  $\mu = \rho\nu$

Therefore, at  $\xi = 2$ ,

$$\frac{\Delta p_w}{\rho} \left( \frac{p_x(o, o)}{\rho} \right)^2 = \frac{1}{9} \nu^2 \omega_w^4(a) \quad (22)$$

In (22),  $\Delta p_w$  is the pressure rise:  $p(0, 0) - p(a, 0)$ ,

with  $x = 0$  : point of separation

$x = a$ : point where  $\omega_w = \omega(0, y_o)$  (close to the point on the wall where  $p_x = 0$ ,  $p = \min$ ).

$p_x = p_x(0, 0)$ : pressure gradient at point of separation.

Equation (22) is essentially Stratford's criterion<sup>5,6</sup>, with a slightly different interpretation of the constant term.

#### IV. Separation Condition for Symmetry Plane of Three-Dimensional Plane Flow

On the symmetry plane ( $z = 0$ ), the transverse velocity component  $w$  is zero, or

$$\left. \begin{aligned} w &= zW(x,y) + O(z^3) \\ H &= p = \frac{1}{2}\rho(u^2 + v^2) \end{aligned} \right\} \quad (23)$$

The Navier-Stokes equations take the form

$$\left. \begin{aligned} H_x &= \rho v \omega_3 + \mu \nabla^2 u \\ H_y &= -\rho u \omega_3 + \mu \nabla^2 v \end{aligned} \right\} \quad (24)$$

where  $\omega_3$  is the vorticity component normal to the symmetry plane and

$$\left. \begin{aligned} \nabla^2 u &= u_{xx} + u_{yy} + u_{zz} = \omega_{2x} - \omega_{3y} \\ \nabla^2 v &= v_{xx} + v_{yy} + v_{zz} = \omega_{3x} - \omega_{1z} \end{aligned} \right\} \quad (25)$$

with  $\omega_1$  and  $\omega_2$  the vorticity components in the  $x, y$  directions, respectively. Let

$$\Psi(x, y) = - \int_0^y u(x, y_1) dy_1 \quad (26)$$

Then  $\Psi_y = -u(x, y)$

$$\text{and } \Psi_x = v(x, y) + \int_0^y W(x, y_1) dy_1 \quad (27)$$

It is again possible to consider the variation  $dH$  on a curve  $y = q(x)$  in the symmetry plane. The counterpart of Eq. 51 becomes

$$dH = \rho\omega_3 \left\{ \Psi_x + q'(x)\Psi_y - \int_0^y W(x, y_1) dy_1 \right\} dx + \mu (\nabla^2 u + q'(x)\nabla^2 v) dx \quad (28)$$

Again let  $q(x)$  be a closed curve composed of the segments  $C$ ,  $L_1$ ,  $L_2$  and again let  $C$  be a contour curve on which  $\omega_3 = \text{const}$ . Evaluating the integral contributions  $\int dH$  for the segments in the same way as before results in

$$\left( \Delta p_w + \rho\omega_3(a, 0) \int_a^0 dx \int_0^{g(x)} W(x, y_1) dy_1 \right) = \rho\omega_3(a, 0)\Psi(0, y_0) - \frac{\rho u^2(0, y_0)}{2} \quad (29)$$

On comparison with Eq. (13), one notes that the counterpart Eq. (29) retains the form of the previous result, requiring but a slight re-interpretation of the pressure-rise term. It is now supplemented by an additional term that accounts for the presence of the transverse velocity. It will be noted that the integral of  $W$  makes it in effect the mean value of  $W$  over the area described by the closed curve  $q(x)$ . As to be expected, if the mean of the transverse velocity is directed toward the symmetry plane, a smaller pressure rise than that needed to separate the original two-dimensional flow is required to separate the flow in the symmetry plane. If the mean of the transverse velocity is directed outward,

the opposite is true. The remainder of the analysis proceeds as before. The separation condition Eq. (22) will be reproduced, with the left-hand side of (29) replacing  $\Delta p_W$ .

### Conclusions

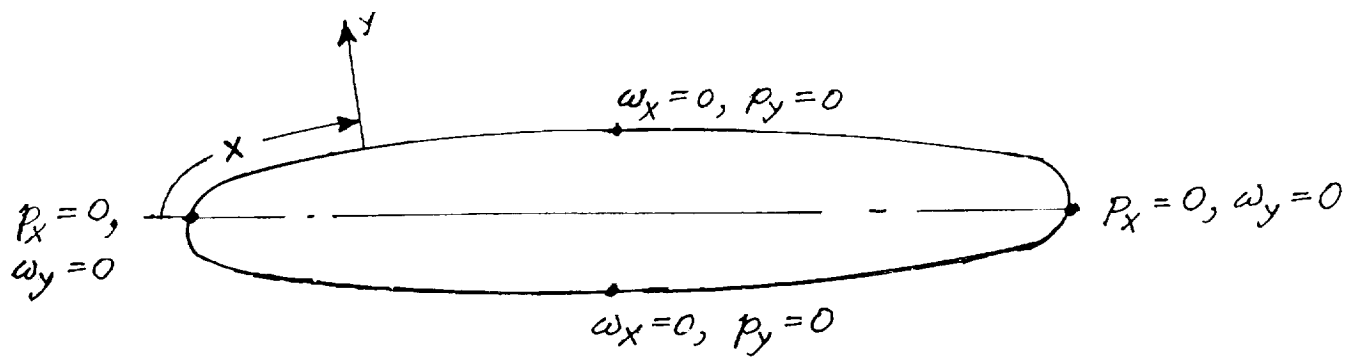
Topological concepts have been used to derive separation conditions for two- and three-dimensional laminar flows. The result for two-dimensional flow confirms the form of the well-known Stratford criterion, giving it a topological basis. The result for three-dimensional flow, applicable to the symmetry plane, is a simple modification of the Stratford form. The pressure-rise term is supplemented by a term that accounts for the presence of a transverse velocity. If the transverse velocity is inward toward the symmetry plane, a smaller pressure rise than that needed to separate a two-dimensional flow is required to separate the flow in the symmetry plane. If the transverse velocity is outward, the opposite is true.

## References

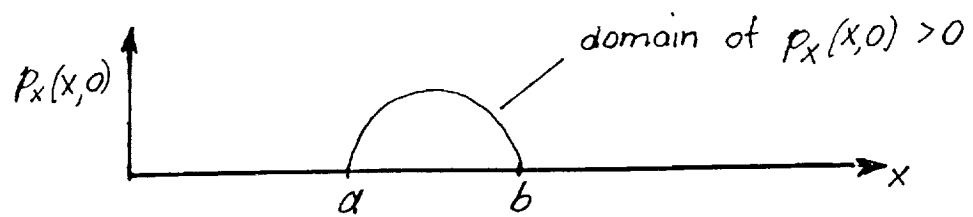
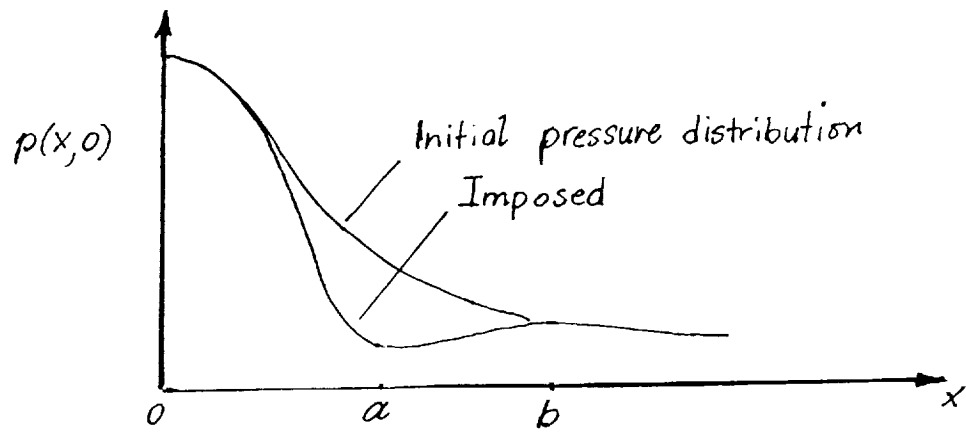
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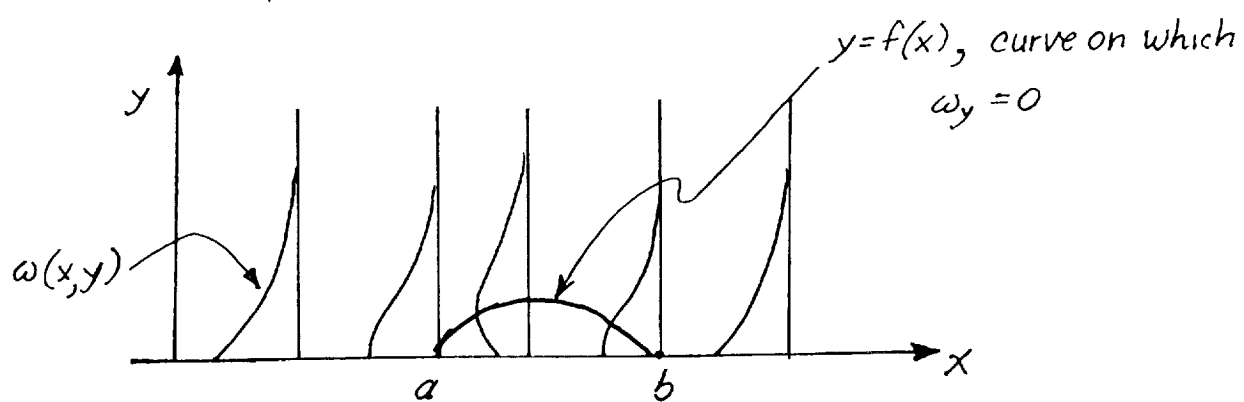
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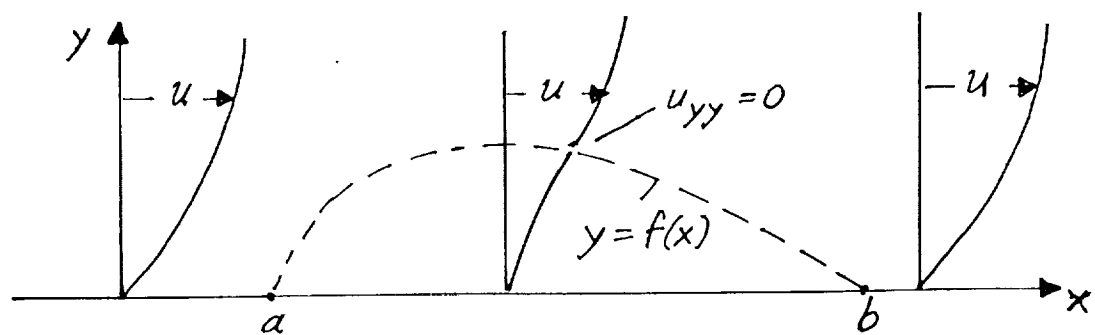


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Figure 1

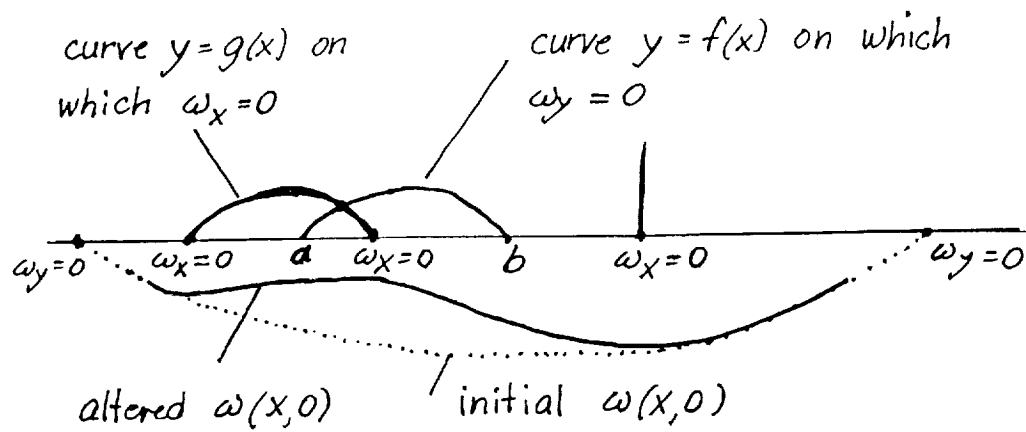




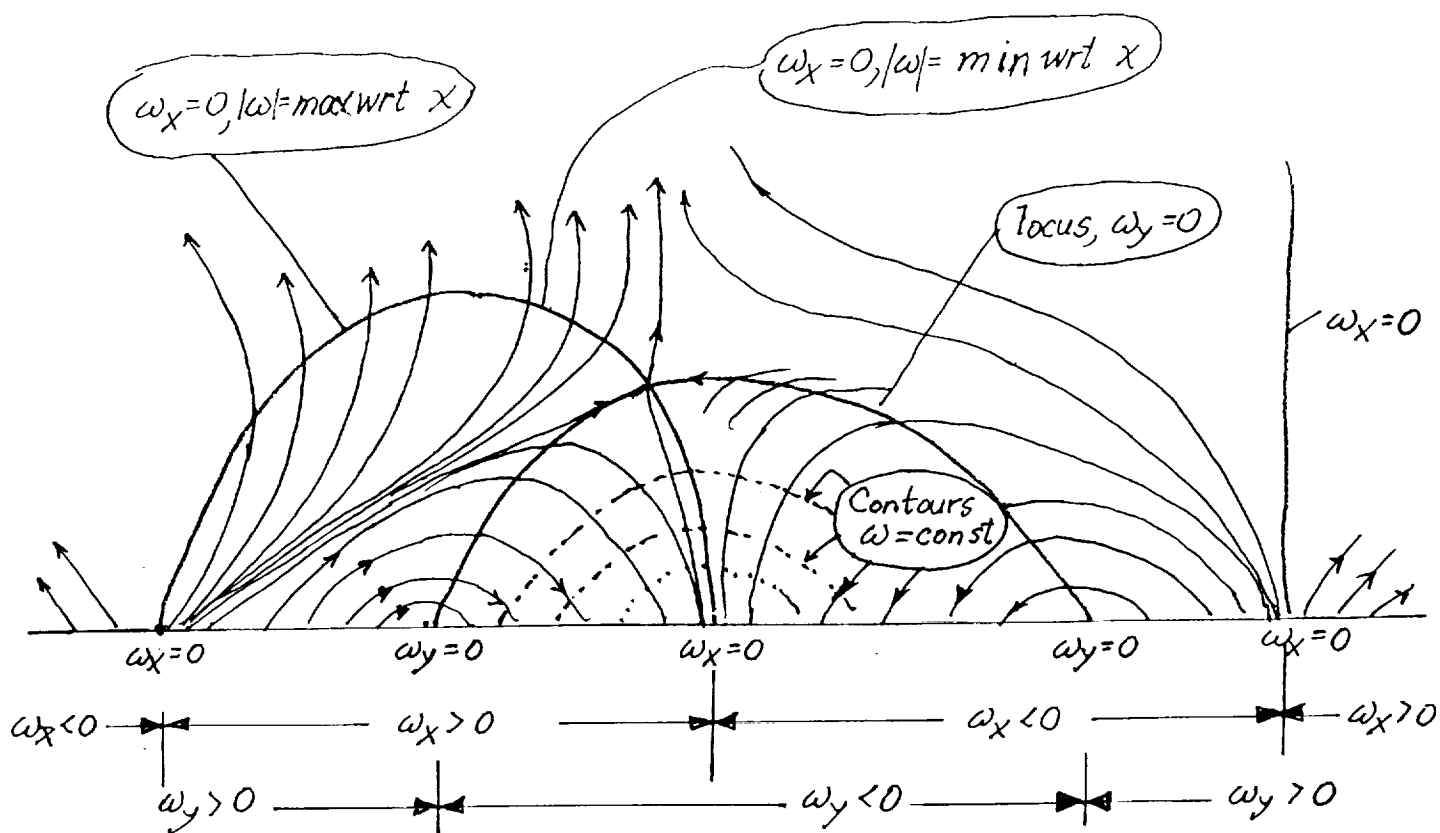
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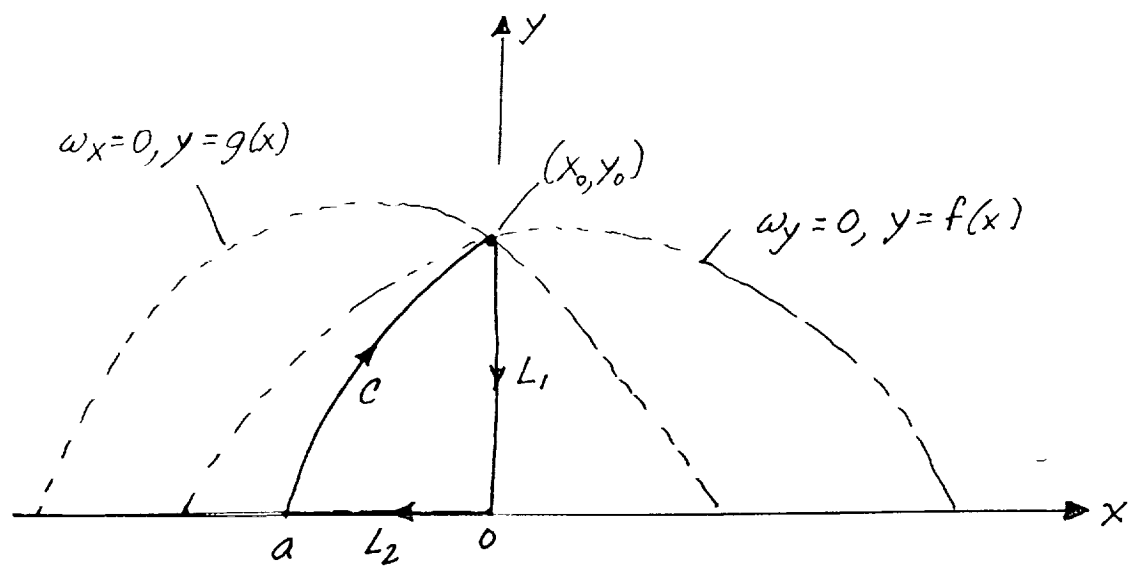
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